**Ch 8 - The Logic of Conditionals**

* the most common valid proof step involving → is called **modus ponens,** aka **conditional elimination**
  + modus ponens is thus an inference rule
* another inference rule is **biconditional elimination**
  + from P and either P ⟺ Q or Q ⟺ P, infer Q
* there are useful equivalences involving the conditionals
  + **Law of Contraposition**
    - P → Q logically equivalent to ~Q → ~P (the contrapositive of the original conditional)
    - it is often easier to prove the contrapositive of the conditional than the conditional itself

**Method of Conditional Proof**

* method that allows you to prove a conditional statement
* suppose you want to prove P → Q
  + temporarily assume the antecedent is true
  + with the original premises plus the additional assumption, if you can prove Q, then you can infer that P → Q follows from the original premises

**example**

**Even(n^2)** → **Even(n)**

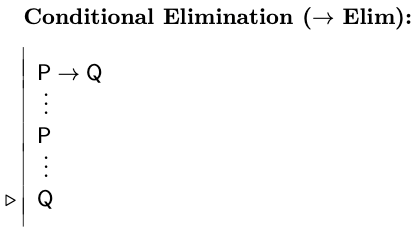
Assume n^2 is even.

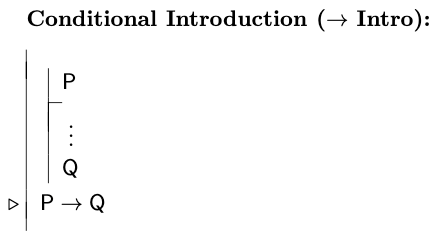
Assume n is odd. Then n^2 = (2m+1)^2. (...) = 2(2m^2+2m) + 1. Thus, n^2 is odd. This statement contradicts the assumption that n^2 is even. Thus, since n^2 being odd is a logical consequence of n being odd, and n^2 being odd is false, then n being odd is false.

Therefore, by negation introduction, n is even.

Since we assumed n^2 is even, and showed that n even follows, we can infer the initial conditional.

**Formal Rules for the Conditionals**

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**Soundness**

* we intend our formal system F to be a correct system of deduction in the sense that any argument that can be proven valid in F should be genuinely valid
  + does F allow us to construct proofs only of genuinely valid arguments?
    - this is the **soundness question for the deductive system F**
* there is a certain vagueness about the notion of logical consequence
  + tautological consequence was introduced as a precise approximation of the informal notion
  + do the rules for the truth-functional connectives allow us to prove only arguments that are tautologically valid?
* Ft is a new notation used to refer to the portion of our deductive system that contains
  + introduction and elimination rules for not, or, and, material conditional, material biconditional, contradiction
* P1,...,Pn ⊢t S means there is a formal proof in Ft of S from premises P1,...,Pn

**Theorem (Soundness of Ft)** If P1, …, Pn ⊢t S then S is a tautological consequence of P1,...,Pn

* in other words, if there is a proof using the connectives in Ft, from P1,...,Pn to S, then S is a tautological consequence of these premises
* the deductive system only allows you to prove valid arguments

**Proof**

* suppose **p** is a proof constructed in the system Ft
* we will show that **any sentence that occurs at any step in p is a tautological consequence of the assumptions in force at that step**

**Assume there is a step in *p* containing a sentence that is not a tautological consequence of the assumptions in force at that step**; call this an **invalid step**

* any step in a proof p is the result of applying one of the twelve inference rules in Ft
* ie, the rule that could result in the invalid step is one of the twelve, ie a disjunction of the twelve, so we can apply an analysis by cases here
* the twelve (propositional) rules of Ft are
  + conjunction intro and elim
  + disjunction intro and elim
  + negation intro and elim
  + contradiction intro and elim
  + conditional intro and elim
  + biconditional intro and elim

**case 1:** → **Intro**

* **assume the first invalid step derives the sentence Q → R from an application of → Intro to an earlier subproof with assumption Q and conclusion R**
* let A1,...,Ak be the assumptions in force at Q → R.
* the assumptions in force at R are A1,...,Ak and Q
* R comes earlier in the proof than the first invalid step, so it is a tautological consequence of A1,...,Ak and Q
* imagine constructing a joint truth table for A1,..., Ak, Q, R, Q → R
* there must be some row in which A1, …, Ak are all true, but Q → R is false, by the assumption of our indirect proof that this step is invalid. Q → R being false can only happen if Q is true and R is false at this step. But at this point the subproof has already happened that showed that when A1, …, Ak are true and Q is true, then R is a tautological consequence of these assumptions and is thus true.
* R must therefore be true at the step with Q → R
* we have reached a contradiction

**case 2:** → **Elim**

* **assume the first invalid step derives the sentence Q from the application of → Elim to two statements that are true at the point in the proof at which the invalid step is derived: P and P → Q.**
* let A1, …, Ak be assumptions in force when Q appears in the proof
* P and P → Q are both valid steps, and are cited by the step Q; since Ft only allows a step to cite steps that occurred before it, P and P → Q occur before the step Q. The assumptions in force at step P and at step P → Q are thus a subset of A1,...,Ak
* construct a joint truth table of A1, …, Ak, P, P → Q, and Q
* there must be a row where Q is false, and yet A1, …, Ak are all true, P and P → Q are therefore true. There must be this row, because otherwise Q would be a tautological consequence of the assumptions in force when it appears in the proof
* if Q is false, then P is false by the law of the contrapositive, but P has already appeared in the proof, so we have a contradiction.

**case 3: ⊥ Elim**

* assume the first invalid step derives the sentence S from a contradiction step
* the premises in force at the contradiction step are also in force at the first invalid step.
* because a contradiction follows from them, the premises are inconsistent.
  + since the premises can never be true all at the same time, the first invalid step is “vacuously” a tautological consequence of the premises.

**case 4: ⊥ Intro**

* assume the first invalid step derives **⊥** from a set of mutually inconsistent sentences that have appeared in the proof up to that point, including the premises A1, …, Ak.
* build a truth table with all the premises and the sentences that have appeared up to that point
* since it is an invalid step, there must be a row in the truth table where all these premises and sentences are true, and the invalid step is false
* but by assumption, these premises and sentences cannot all be true simultaneously, since they are inconsistent

**Given that all cases result in a contradiction, by disjunction elimination, we can assert ⊥. Ie, our initial assumption that there is a step in p containing a sentence that is not a tautological consequence of the assumptions in force at that step is false. There can be no such invalid step. All steps are therefore tautological consequences of assumptions in force at that step.**

* therefore, whatever we prove is always a tautological consequence of the premises and assumptions up to the point of the proved statement
* We cannot prove something that is not a tautological consequence of the premises and assumptions, ie something that can be false when the premises and assumptions are true.

**Definition (Corollary):** result which follows with little effort from an earlier theorem.

**Corollary:** If ⊢t S, that is, if there is a proof of S in Ft with no premises, then S is a tautology.

**Completeness**

* Given any premises P1, …, Pn and any tautological consequence S of these premises, does our deductive system Ft allows us to construct a proof of S from P1,...,Pn?

**Theorem (Completeness of Ft):** If a sentence S is a tautological consequence of P1,..., Pn then P1,...,Pn ⊢t S

* this result is called the **Completeness Theorem** because it tells us that the intro and elim rules are complete for the logic of the truth-functional connectives: **anything that is a logical consequence simply in virtue of the meanings of the truth-functional connectives can be proven in Ft**
* **thus, all tautologies (and tautologically valid arguments) are provable in Ft**
* both theorems have to do with tautological consequence, not logical consequence
* soundness of Ft does not tell us that it is possible to prove a statement that is a logical consequence of some premises
* depending on the statements in question, we need certain additional rules such as identity or rules having to do with particular predicates

Recap

* Given premises P1,...,Pn and a conclusion S, if the argument formed by these statements is valid, then S is a logical consequence of P1,..., Pn: it is impossible for the premises to be true and S to be false. This relationship is expressed by the connective material conditional: P1 & … & Pn → S.
* This statement also has a truth value. If S is a logical consequence of P1, …, Pn then the row in the truth table where the premises are false and S is true (and thus the material conditional sentence is false) is not logically possible. Thus the conditional is always true in every logically possible circumstance. Thus it is a logical truth.
* Now, why do we know that S is a logical consequence of P1, …, Pn? By utilizing a proof.
* The Soundness Theorem tells us that if there is a proof of S from premises P1, …, Pn then S is a tautological consequence of P1, …, Pn. This means that if we build a truth table of P1, …, Pn then the truth values of S coincide with the truth values of P1 & … & Pn. In particular, whenever the premises are true, S is true.
* If there are no premises, and instead we have the statement P1 & … & Pn → S, which represents an argument, and we can prove this statement from no premises, then the statement is a tautology: it is always true.
* If we can show that a statement is not a tautological consequence of certain premises, then we can also conclude that there is no proof of the statement from the premises. This is a simple application of the Law of Contrapositive to the Soundness Theorem.
* The Completeness Theorem tells us that if a statement S is a tautological consequence of a set of premises P1, …, Pn, then there is a proof of this relationship within Ft.
* If we can show that a statement S is a tautological consequence of some premises, then we know the proof exists.
* If we can show that S is not a tautological consequence of some premises, then we know there is no proof of S from the premises.

**Taut Con procedure**

* checks whether a sentence is a tautological consequence of whatever sentences are cited in support
* if it is, then we know there is a proof (Completeness Theorem)
* if it is not, then we know there is not a proof (Soundness Theorem)